Aperiodic Subshifts of Finite Type on Groups

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Abstract

In this note we prove the following results:

- If a finitely presented group G admits a strongly aperiodic SFT, then G has decidable word problem. More generally, for f.g. groups that are not recursively presented, there exists a computable obstruction for them to admit strongly aperiodic SFTs.
- On the positive side, we build strongly aperiodic SFTs on some new classes of groups. We show in particular that some particular monster groups admits strongly aperiodic SFTs for trivial reasons. Then, for a large class of group G, we show how to build strongly aperiodic SFTs over $\mathbb{Z} \times G$. In particular, this is true for the free group with 2 generators, Thompson's groups T and V, $PSL_2(\mathbb{Z})$ and any f.g. group of rational matrices which is bounded.

While Symbolic Dynamics [LM95] usually studies subshifts on $\mathbb{Z}$, there has been a lot of work generalizing these results to other groups, from dynamicians and computer scientists working in higher dimensions ($\mathbb{Z}^d$ [Lin04]) to group theorists interested in characterizing group properties in terms of topological or dynamical properties [CSC10].

In this note, we are interested in the existence of aperiodic Subshifts of Finite Type (SFT for short), or more generally of aperiodic effectively closed shifts.

A subshift on a group G corresponds informally to a way of coloring the elements of the group, subject to some local constraints. The constraints are in finite number for a SFT, the constraints are given by an algorithm for an effectively closed shift.

Of great interest is the existence of aperiodic subshifts, which are nonempty subshifts for which no coloring has a translation vector, i.e. is invariant under translation by a nonzero element of G.

An important example of a group with an aperiodic SFT is the group $\mathbb{Z}^2$, for which SFTs are sometimes called tilings of the plane and given by Wang tiles [Wan61], the most famous example being the Robinson tiling [Rob71].

Not all group admits an aperiodic SFTs though, it is for example easy to see that there are no aperiodic SFT over $\mathbb{Z}$. However, all (countable) groups admits aperiodic shifts, this result is surprisingly nontrivial and quite recent [GJS09].

There has been a lot of work proving how to build aperiodic SFTs in a large class of groups, and more generally tilings on manifolds [Moz97, Coh14]. It

is an open question to characterize groups that admit strongly aperiodic SFTs. Cohen[Coh14] proved that this property is a quasiisometry invariant, and Carroll and Penland [CP] proved it is a commensurability invariant.

In the first part of this article, we use computability theory to prove that groups admitting aperiodic SFTs must satisfy some computability obstruction. In particular, a finitely presented group with an aperiodic SFT has a decidable word problem. This is proven in section 2. Cohen[Coh14] showed that f.g. groups admitting strongly aperiodic SFTs are one ended and asked whether it is a sufficient condition. Our first result proves in particular that this condition is not sufficient.

For f.g. groups that are not finitely presented, the computability condition is harder to understand: Intuitively it means that the information about which products of generators are equal to the identity is enough to know which products aren't, i.e. we can obtain negative information about the word problem from positive information. The exact criterion is formulated precisely using the concept of enumeration reducibility. This generalization is presented in section 3, and might be omitted by any reader not familiar or interested with recursion theory.

The more general result we obtain is as follows:

Theorem. *If a f.g. group G admits a normally aperiodic effectively closed subshift, then the complement of the word problem of G is enumeration reducible to the word problem of G.*

For finitely presented groups, this implies the word problem of G is decidable.

Normal aperiodicity is a weakening of the notion of aperiodicity, which is intermediate between the notion of weak aperiodicity and of strong aperiodicity. Strongly aperiodic subshifts ask that the stabilizer of each point is finite. Here we ask that the stabilizer of each point does not contain a normal subgroup.

The first part of the article is organized as follows. Subshifts can be defined as shift-invariant topologically closed sets on A^G, the sets of functions from G to A. The first section introduces an effective notion of closed sets on $A^{\mathbb{F}_p}$ and then on A^G, and proves some link between the two. In the second section, we use this effective notion to prove the main result for recursively presented groups In the third section, we use concepts of computability theory to generalize the results to any f.g. group.

In the second part of the article, we exhibit aperiodic subshifts of finite type for some groups G. We first show that some monster groups, which are infinite simple groups with some bad properties, admits strongly aperiodic SFTs The SFT we obtain are quite trivial and degenerate.

In the last section, we will remark how a variation on a technique by Kari gives aperiodic SFTs on $\mathbb{Z} \times G$ for a large class of group G. We do not know if there exists an easier proof of this statement. This class of groups contains the free group, f.g. subgroups of $SO_n(\mathbb{Q})$, and Thompson's group T and V.

1 Effectively closed sets on Groups

We first give definitions of effectively closed sets, which are some particular closed subsets of the Cantor Space A^G and $A^{\mathbb{F}_p}$. The reader fluent with symbolic dynamics should remark that the sets we consider are not supposed to be translation(shift)-invariant in this section.

1.1 Effectively closed sets on the free group

Let $\mathbb{F}_p$ denote the free group on p generators, with generators $x_1 \dots x_p$. Unless specified otherwise, the identity on $\mathbb{F}_p$, and any other group will be denoted by λ, and the symbol 1 will be used only for denoting a number. Let A be a finite alphabet.

A *word* is a map w from a finite part of $\mathbb{F}_p$ to A. A configuration $x \in A^{\mathbb{F}_p}$ *disagrees* with a word w if there exists g so that $x_g \neq w_g$ and both sides are well defined.

Definition 1.1. *Let L be a list of words. The* closed set *defined by L is the subset $S_L^{\mathbb{F}_p}$ of $A^{\mathbb{F}_p}$ of all configurations x that disagree with all words in L.*

Such a set is a closed set for the prodiscrete topology on $A^{\mathbb{F}_p}$.

if S is a closed set of $A^{\mathbb{F}_p}$, we denote by $\mathcal{L}(S)$ the set of all words of $\mathbb{F}_p$ that disagree with S. Note that $S = S_{\mathcal{L}(S)}$.

Definition 1.2. *A closed set S of $A^{\mathbb{F}_p}$ is effectively closed if $S = S_L$ for a recursively enumerable set of words L. Equivalently, $\mathcal{L}(S)$ is recursively enumerable.*

The equivalence needs a proof. We give it in the form of the following result, which will be useful later on.

Proposition 1.3. *There exists an algorithm that, given an effective enumeration L, halts iff $S_L^{\mathbb{F}_p}$ is empty.*

Proof. For a finite set L, it is easy to test if $S_L^{\mathbb{F}_p}$ is empty: just test all possible words of $A^{\mathbb{F}_p}$ defined on the union of the supports of all words in L.

Furthermore, by compactness, for an infinite L', $S_{L'}^{\mathbb{F}_p}$ is empty iff there exists a finite $L \subseteq L'$ so that $S_L^{\mathbb{F}_p}$ is empty.

Now, if L is effective, consider the following algorithm: enumerate all elements w_i in L, and test at each step if $S_{\{w_1,\dots,w_n\}}^{\mathbb{F}_p}$ is empty. By the first remark, it is indeed an algorithm. By the second remark, this algorithm halts iff $S_L^{\mathbb{F}_p}$ is empty. □

Proof of the equivalence in the definition. Suppose that $S = S_L^{\mathbb{F}_p}$ is effectively closed. We will prove that $\mathcal{L}(S)$ is recursively enumerable. Let w be a finite word. Suppose that w is defined over I. Let W be the set of all words defined over I that are incompatible with w. Then $w \in \mathcal{L}(S)$ iff $S_{L \cup W} = \emptyset$.

□

1.2 Effectively closed sets on groups

We will now look at closed sets of A^G. From now on, a given finitely generated group G will always be given as a quotient of a free group, that is $G = \mathbb{F}_p/R$ for R a normal subgroup of $\mathbb{F}_p$. As long as G is finitely generated, it is routine to show that all such representations give the exact same definition of effectiveness.

Let ϕ be the natural map from $\mathbb{F}_p$ to G. For $g, h \in \mathbb{F}_p$ we will write $g =_G h$ for $\phi(g) = \phi(h)$.

Let w be a word on $\mathbb{F}_p$. We say that w is a G-word if $w_g = w_h$ whenever $g =_G h$ and both sides are defined.

A configuration $x \in A^G$ *disagrees* with a word w if there exists $g \in \mathbb{F}_p$ so that $x_{\phi(g)} \neq w_g$ and both sides are defined. Note that a configuration in A^G always disagrees with a word which is not a G-word.

Definition 1.4. *The closed set on G defined by L is the subset S_L^G of A^G of all configurations x that disagree with all words in L.*

Such a set is a closed set for the prodiscrete topology on A^G.

If S is a closed set of A^G, we denote by $\mathcal{L}(S)$ the set of all words of $\mathbb{F}_p$ that disagree with S. Note that $S = S_{\mathcal{L}(S)}$.

Note in particular that all words that are not G-words are always elements of $\mathcal{L}(S)$.

Definition 1.5. *A closed set S of A^G is effectively closed if $S = S_L^G$ for a recursively enumerable set of words L.*

Note that this is no longer equivalent to $\mathcal{L}(S)$ being recursively enumerable. Indeed, $S = A^G$ is effectively closed but $\mathcal{L}(A^G)$ is recursively enumerable only if G is recursively presented (see below).

In the following we will see closed sets of A^G as closed sets of $A^{\mathbb{F}_p}$.

To do so, denote by Per_G the set of all configurations of $A^{\mathbb{F}_p}$ which are G-consistent, that is $x_g = x_h$ whenever $g =_G h$. It is clear that Per_G is a closed set: In fact $\mathcal{L}(Per_G)$ is exactly the set of words that are not G-words.

Furthermore Per_G is isomorphic to A^G: There is a natural map ψ from Per_G to A^G defined by $\psi(x) = y$ where $y_{\phi(g)} = x_g$. ψ is invertible with inverse defined by $\psi^{-1}(y) = x$ where $x_g = y_{\phi(g)}$.

This bijection preserves the closed sets in the following sense:

Fact 1.6.

$$\psi(S_L^{\mathbb{F}_p} \cap Per_G) = S_L^G$$

$$\mathcal{L}(S_L^{\mathbb{F}_p} \cap Per_G) = \mathcal{L}(S_L^G)$$

While $S_L^{\mathbb{F}_p}$ is always effective if L can be enumerated, it might be possible for $S_L^{\mathbb{F}_p} \cap Per_G$ to not be effective. In fact:

Proposition 1.7. *Let A be an alphabet of size at least 2.*

Per_G is effective iff G has a recognizable word problem.

A recognizable word problem means that there is an algorithm that, given a word w in $\mathbb{F}_p$, halts iff $w =_G \lambda$. This is equivalent to saying that G is recursively presented.

Proof. If G has a recognizable word problem, we can enumerate all pairs $(g,h) \in \mathbb{F}_p$ s.t. $g =_R h$, and thus enumerate the set of words that are not G-words, that is $\mathcal{L}(Per_G)$.

Conversely, suppose that $\mathcal{L}(Per_G)$ is enumerable. Let $\{a,b\}$ be some fixed letters from A, with $a \neq b$. For $g \in G$, consider the word w over $\{\lambda, g\}$ defined by $w_\lambda = a, w_g = b$. Then $g =_G \lambda$ iff $w \in \mathcal{L}(Per_G)$. □

Corollary 1.8. *If G has a recognizable word problem and S_L^G is effectively closed, then $S_L^{\mathbb{F}_p} \cap Per_G$ is effectively closed.*

2 Effective subshifts on Groups

If $x \in A^G$, denote by gx the configuration of A^G defined by $(gx)_h = x_{g^{-1}h}$. This defines an action of G on A^G.

Definition 2.1. *A closed set X of A^G is said to be a subshift if $x \in X, g \in G$ implies that $gx \in X$.*

X is an effectively closed subshift if X is effectively closed and is a subshift.

X is a SFT if there exists a finite set L so that $X = S_{\{g^{-1}w, g \in G, w \in L\}} = S_{\{g^{-1}w, g \in \mathbb{F}_p, w \in L\}}$. In particular a SFT is always effectively closed.

Fact 2.2. *If X is a subshift, $\psi^{-1}(X) \subseteq Per_G$ is a subshift.*

Hence any subshift of A^G lifts up to a subshift of $A^{\mathbb{F}_p}$.

As a warmup to the theorems, we consider $X_{\leq 1}$, the subset of $\{0,1\}^G$ of configurations that contains at most one symbol 1. It is easy to see that $X_{\leq 1}$ is closed, and a subshift.

Proposition 2.3. *Suppose that G has a recognizable word problem.*

If $X_{\leq 1}$ is effectively closed then the word problem on G is decidable.

Proof. $X_{\leq 1}$ lifts up to a subshift Y on $\mathbb{F}_p$ with the property that (a) Y is effective (as G is recursively presented), hence $\mathcal{L}(Y)$ is enumerable (b) Y consists of all configurations so that $x_g = x_h = 1 \implies g =_G h$.

Now let $g \in \mathbb{F}_p$. Let w be the word defined by $w_\lambda = 1$ and $w_g = 1$. Then $w \in \mathcal{L}(Y)$ iff $g \neq_G \lambda$.

The complement of the word problem is recognizable, therefore decidable. □

In the following, we are now interested in aperiodic subshifts.

Definition 2.4. *For $x \in A^G$ denote by $Stab(x) = \{g | gx = x\}$.*

A (nonempty) subshift X is strongly aperiodic iff for every $x \in X$, $Stab(x)$ is finite.

A (nonempty) subshift X is normally aperiodic iff for every $x \in X$, $\cap_{h \in G} Stab(hx)$ is finite.

Note that there are conflicting definitions in the literature for strong aperiodicity: Some require that $Stab(x) = \{\lambda\}$ for all $x \in X$. The only result in this paper where this makes a difference is Prop 4.2.

Both properties are equivalent for commutative groups. $\cap_{h\in G} Stab(hx)$ will be called the *normal stabilizer* of x. It is indeed a normal subgroup of G, and the union of all normal subgroups of $Stab(x)$.

Our first result states that a strongly aperiodic effectively closed subshift (and in particular a strongly aperiodic SFT) forces the group to have a decidable word problem in the class of torsion-free recursively presented group. The next proposition strengthens the result by deleting the torsion-freeness requirement.

Proposition 2.5. *Let G be a torsion-free recursively presented group.*

If G admits a strongly aperiodic effective subshift, then G has decidable word problem.

Proof. Let X be the strongly aperiodic effective subshift. X lifts up to a subshift Y on $A^{\mathbb{F}_p}$.

Let $R = \{g | g =_G \lambda\}$.

Note that if $\psi(y) = x$, then $Stab(y) = Stab(x)R = RStab(x)$. Furthermore, if G is torsion-free, then $Stab(x) = \{\lambda\}$, hence for all $y \in Y$, $Stab(y) = R$.

Now let $g \in \mathbb{F}_p$. Let $Z = \{x | \forall t, x_{gt} = x_t\} = \{x \in A^{\mathbb{F}_p} | g \in Stab(x)\}$.

It is easy to see that Z is effective. Furthermore $Y \cap Z = \emptyset$ iff $g \neq_G \lambda$.

As there is a semialgorithm to test whether $Y \cap Z = \emptyset$, we get that the complement of the word problem is recognizable, hence decidable. ☐

Proposition 2.6. *Let G be a recursively presented group.*

If G admits a normally aperiodic effectively closed subshift, then it admits a normally aperiodic effectively closed subshift X where for all $x \in X$, $\cap_{h\in G} Stab(hx) = \lambda$.

Proof. Let X be normally aperiodic.

The proof is in two steps. In the first step, we will prove that there exists a finite normal subgroup H of G and a nonempty effective subshift Y so that for all $x \in Y$, $\cap_{g\in G} Stab(gx) = H$.

Let $H_0 = \{\lambda\}$. Suppose that there exists $x \in X$ so that $H_0 \subsetneq \cap_{h\in G} Stab(hx)$. Then let's denote $H_1 \supset H_0$ the normal subgroup on the right.

We do the same for H_1, building progressively a chain of normal subgroups $H_1 \dots H_n \dots$.

It is impossible however to obtain an infinite chain this way. Indeed, as for all i, there exists x_i so that $\cap_{g\in G} Stab(gx_i) = H_i$, a limit point x of x_i would verify $\cap_{g\in G} Stab(gx) \supseteq \cup_i H_i$, hence x would be a configuration with an infinite normal stabilizer, impossible by definition.

Hence this process will stop, and we obtain some finite normal subgroup H of G and a point x_0 so that $\cap_{h\in G} Stab(hx) = H$ and no point x has a larger normal stabilizer.

Now let $Y = \{x \in X | \forall g \in G, h \in H, hgx = gx\}$. Y is nonempty, as it contains x_0. As H is finite, Y is clearly effectively closed. As H is normal, it is a subshift. Furthermore, for all $x \in Y$, $\cap_{h\in G} Stab(hx) = H$.

Now the second step. Take $Z = \{x \in H^G | \forall g \in G, h \in H \setminus \{\lambda\}, x_{hg} \neq x_g\}$. Z is clearly effectively closed. As H is normal, it is a subshift[1]. Z is also nonempty: Write $G = HI$ where I is a family of representatives of G/H. Then the point

[1] Indeed, let $z \in Z$ and $t \in G$. Let $g \in G$ and $h \in H \setminus \{\lambda\}$. Then $(tz)_{hg} = z_{t^{-1}hg} = z_{t^{-1}htt^{-1}g} \neq z_{t^{-1}g} = (tz)_g$, hence $tz \in Z$

z defined by $z_g = h$ if $g \in hI$ is in Z, hence Z is nonempty[2]. Furthermore, if $z \in Z$, then $Stab(x) \cap H = \{\lambda\}$ [3]. As a consequence, $Z \times Y$ is a nonempty subshift for which for all $x \in Z \times Y$, $\cap_{h \in G} Stab(hx) = \{\lambda\}$. □

Theorem 1. *Let G be a recursively presented group.*

If G admits a normally aperiodic effectively closed subshift, G has decidable word problem.

Proof. This is more or less the same proof as before, with one slight difference.

Let X be the normally aperiodic effectively closed subshift. We may suppose by the previous proposition that for all $x \in X$, $\cap_{h \in G} Stab(hx) = \{\lambda\}$.

X lifts up to a subshift Y on $A^{\mathbb{F}_p}$ with the following property: If $y \in Y$, then $\cap_{h \in \mathbb{F}_p} Stab(hy) = R$.

Now let $g \in \mathbb{F}_p$. Let $Z = \{y | \forall h \in \mathbb{F}_p, ghy = hy\} = \{y | g \in \cap_{h \in \mathbb{F}_p} Stab(hy)\}$. Z is effective. Furthermore $Y \cap Z = \emptyset$ iff $g \neq_G \lambda$.

Emptyness is recognizable, hence the complement of the word problem is recognizable, hence decidable.

□

Corollary 2.7. *Let G be a recursively presented group that admits a strongly aperiodic SFT. Then G has a decidable word problem.*

3 Enumeration degrees

In this section, we generalize the previous results to any finitely generated groups, whose presentation might be not recursive.

Recall that the word problem of G is the set

$$W(G) = \{g \in \mathbb{F}_p | g =_G \lambda\}$$

If G is any f.g. group, there are various ways that a subshift $X = S_L$ could be said to be computable in G:

- We can enumerate a list of forbidden pseudo-words for S
- We can enumerate a list of forbidden pseudo-words for S give an oracle for $W(G)$
- We can enumerate a list of forbidden pseudo-words for S give a list of all elements of $W(G)$

The last two are possibly different: In the last case, we are only given access to positive informations, i.e. the elements that are in $W(G)$, not those that aren't.

The definitions and results below involve the notion of *enumeration reducibility* [FR59, Odi99].

Enumeration degrees, and enumeration reducibility, is a notion from computability theory that is quite natural in the context of presented groups and subshifts, as it captures (in computable terms) the fact that the only information we have about these objects are positive (or negative) information: In a

[2] Indeed, let $g \in G$ and $h \in H \setminus \{\lambda\}$. Then $z_g = k$ where $g \in kI$ for some $k \in H$. But $hg \in (hk)I$ hence $z_{hg} = hk \neq k = z_g$.

[3] Indeed, for $h \in H \setminus \{\lambda\}$, $(hx)_\lambda = x_{h^{-1}} \neq x_\lambda$, hence $h \notin Stab(x)$.

subshift (effective or not), we usually have ways to describe patterns that do not appear, but no procedure to list patterns that appear. In a presented group, we have information about elements that correspond to the identity element of the group, but no easy way to prove that an element is different from the identity.

This reduction has been used already in the context of groups [HS88, Chapter 6] or in symbolic dynamics [AS09].

3.1 Definitions

If A and B are two sets of numbers (or words in $\mathbb{F}_p$), we say that A is enumeration reducible to B if there exists an algorithm that produces an enumeration of A from any enumeration of B. Formally:

Definition 3.1. *A is enumeration reducible to B, written $A \leq_e B$, if there exists a partial computable function f that associates to each (n, i) a finite set $D_{n,i}$ s.t. $n \in A \iff \exists i, D_{n,i} \subseteq B$.*

We will first give here a few easy facts, and then examples relevant to group theory and symbolic dynamics.

Fact 3.2. *A is recursively enumerable iff $A \leq_e \emptyset$.*

In particular, if A is recursively enumerable, then $A \leq_e B$ for all B.

$A \leq_e B \oplus \overline{B}$ iff A is enumerable given B as an oracle.

$B \oplus \overline{B}$ denotes the set $\{(1, x) | x \in B\} \cup \{(1, x) | x \notin B\}$.

Here are some examples relevant to group theory:

Fact 3.3. *(Formal version) Let $G = \langle X|R \rangle$ be a finitely generated group, with $R \subseteq \mathbb{F}_p$ and N be the normal subgroup of $\mathbb{F}_p$ generated by R. Then $N \leq_e R$. In particular, if R is finite, then N (hence the word problem over G) is recursively enumerable.*

(Informal version) From a presentation R of a group, we can list all elements that correspond to the identity element of the group (but in general we cannot list elements that are not identity of the group). In terms of reducibility, the set of all elements that correspond to the identity is the smallest possible presentation of a group.

Indeed $g \in N$ iff there exists $g_1 \dots g_k \in \mathbb{F}_p, u_1 \dots u_k \in R \cup R^{-1}$ so that $g = g_1 u_1 g_1^{-1} g_2 u_2 g_2^{-1} \dots g_k u_k g_k^{-1}$. Given any enumeration of R (and as $\mathbb{F}_p$ is enumerable), we can therefore enumerate N.

Here are some examples relevant to symbolic dynamics or topology.

Fact 3.4. *(Formal version) Let $S = S_L$ be any closed set. Then $\mathcal{L}(S) \leq_e L$.*

In particular if L is enumerable then $\mathcal{L}(S)$ is recursively enumerable.

(Informal version) From any description of a closed set in terms of some forbidden words, we may obtain a list of all words that do not appear (but usually not of patterns that appear). In terms of reducibility, the set of all words that do not appear is the smallest possible description of a closed set.

Subshifts are particular closed sets, so this is also true for subshifts. In particular the set of patterns that do not appear in a SFT (over $\mathbb{Z}$, or $\mathbb{F}_p$) is recursively enumerable.

Proof. This is a straightforward generalization of Prop. 1.3 and the subsequent proof.

Let w be any word, defined over a finite set B. Let W be the set of all words defined over B that are incompatible with w. Then $w \in \mathcal{L}(S)$ iff $S_{L\cup W} = \emptyset$.

Now $S_{L\cup W} = \emptyset$ iff there exists a finite set $L' \subseteq L$ s.t. $S_{L'\cup W} = \emptyset$.

Thus, if $(F(n,w))_{n\in\mathbb{N}}$ is the computable enumeration of all finite sets of words s.t that $S_{F(n,w)\cup W} = \emptyset$, then $w \in \mathcal{L}(S)$ iff $\exists n, F(n,w) \subseteq L$. $\square$

3.2 Generalizations

Now we explain how this concept gives generalizations of the previous theorems.

First, we look at subsets of $A^{\mathbb{F}_p}$ that are effective given an enumeration of B. This definition is nonstandard:

Definition 3.5. *A set $S \subseteq A^{\mathbb{F}_p}$ is* B-enumeration-effective *if $S = S_L$ for some set of words L so that $L \leq_e B$.*

Here are a few examples:

- $\{x \in \{0,1\}^{\mathbb{F}_p} | \forall h \in B, x_h = 1\}$ is B-enumeration effective
- $\{x \in \{0,1\}^{\mathbb{F}_p} | \forall g \in \mathbb{F}_p, \forall h \in B, x_{gh} = x_h\}$ is B-enumeration effective
- $\{x \in \{0,1\}^{\mathbb{F}_p} | \forall h \notin B, x_h = 1\}$ is usually not B-enumeration effective. It is B-enumeration effective iff the complement of B is enumeration reducible to B. It happens for example whenever the complement of B is enumerable, regardless of the status of B.

If we go back to the different ways to define a subshift from a group given at the introduction of this section, they correspond to three different notions of enumeration-effectiveness:

- The first one corresponds to $\emptyset$-enumeration-effective
- The second one corresponds to $W(G) \oplus \overline{W(G)}$-enumeration-effective
- The third one corresponds to $W(G)$-enumeration-effective

The second notion is what is called a G-effective subshift in the vocabulary of [ABS].

Definition 3.6. *We will say that $X \in A^{\mathbb{F}_p}$ is G-enumeration effective whenever X is $W(G)$-enumeration effective.*

Proposition 3.7 (Analog of Prop 1.7)**.** *Let A be an alphabet of size at least* 2. *Then Per_G is G-enumeration effective. More precisely, $\mathcal{L}(Per_G) \equiv_e W(G)$.*

Proposition 3.8. *If X, Y are two closed sets in $A_p^{\mathbb{F}}$, then $\mathcal{L}(X \cap Y) \leq_e \mathcal{L}(X) \cup \mathcal{L}(Y)$*

Proof. $X \cap Y = S_{\mathcal{L}(X)\cup\mathcal{L}(Y)}$. $\square$

Corollary 3.9 (Analog of Cor. 1.8)**.** *If X is an effectively closed set of A^G,then $\mathcal{L}(X) \leq_e W(G)$. More generally, if $X = S_L^G$ then $\mathcal{L}(X) \leq_e L \oplus W(G)$.*

Proof. $\mathcal{L}(X) = \mathcal{L}(S_L^G) = \mathcal{L}(S_L^{\mathbb{F}_p} \cap Per_G) \leq_e L \cup \mathcal{L}(Per_G) \leq_e L \oplus W(G)$. $\square$

Proposition 3.10 (Analog of Prop. 2.3)**.** *If $X_{\leq 1}$ is effectively closed (in particular if it is an SFT) then $\overline{W(G)} \leq_e W(G)$*

Proof. $X_{\leq 1}$ lifts up to a subshift Y of $A^{\mathbb{F}_p}$ which is G-enumeration effective.

By the proof of Prop. 2.3, there exists a uniform family w_g of words so that $g \neq \lambda$ iff $w_g \in \mathcal{L}(X)$. This implies easily that $W(G) \leq_e \mathcal{L}(Y)$. ☐

Theorem 2 (Analog of Th. 1)**.** *If G admits a normally aperiodic effective subshift X, then $\overline{W(G)} \leq_e W(G)$.*

Proof. From the proof of Th. 1, there exists a G-enumeration effective subshift Y on $\mathbb{F}_p$, and a (uniform) family of effectively closed sets X_g so that $Y \cap X_g = \emptyset$ iff $g \neq \lambda$.

Let $(F(n, g))_{n\in\mathbb{N}}$ be a computable enumeration of all finite sets F for which there exists $G \subseteq L_g$ so that $S_{F\cup G} = \emptyset$ (this can indeed be enumerated as L_g can be enumerated).

Then $g \neq_G \lambda$ iff $\exists n F(n, g) \subseteq \mathcal{L}(Y)$, hence $\overline{W(G)} \leq_e \mathcal{L}(Y)$ ☐

Hence the existence of a strongly aperiodic SFT implies that the complement of the word problem can be enumerated from the word problem. This is not a vacuous hypothesis: If G is recursively presented (hence $W(G)$ is recursively enumerable), this implies that $\overline{W(G)}$ is also recursively enumerable, hence recursive.

Another way of stating the result is that, for a group having this property, the notion of G-enumeration effectiveness and G-effectiveness coincide.

4 Aperiodic SFTs on monster groups

In this section we exhibit examples of aperiodic SFTs on some monster groups. All our examples are trivial and somewhat degenerate. Nevertheless these are aperiodic SFT.

Simple groups Our last section introduces a computability obstruction that must satisfy all groups G that admits an aperiodic SFT: The complement of the word problem must be enumeration reducible to the word problem.

There are a few well known algebraic classes of groups that satisfy this property: In particular, all f.g. simple groups have this property.

Proposition 4.1. *If G is a f.g. simple group, then G admits a normally aperiodic SFT*

Proof. Fix some given nontrivial element a of G. And define

$$X = \left\{x \in \{0, 1, 2\}^G \middle| \forall g \in G, (gx)_a \neq (gx)_\lambda\right\}$$

Thus X is an SFT. It is easy to see that X is nonempty.

Now let $x \in X$. Then $(a^{-1}x)_\lambda = x_a \neq x_\lambda$. Thus $a^{-1}x \neq x$, which means the stabilizer of x is not the whole group. A fortiori, the normal stabilizer of x is also not the whole group: $\cap_{h\in G} Stab(hx) \neq G$. As the left term is a normal subgroup of G and G is simple, this implies $\cap_{h\in G} Stab(hx) = \{\lambda\}$. ☐

The proof works verbatim for any group G with no infinite normal subgroups.

Monster groups The previous SFT is only normally aperiodic. For monster groups (which are actually examples of simple groups), we can go further, and obtain strongly aperiodic SFT.

We will use three classes of monster groups:

- Tarski monster groups [Ol'83], for which all nontrivial proper subgroups are of finite order p
- The Osin monster groups [Osi10], for which any two nonzero element are conjugate.
- The Ivanov groups [Ol'91, Th. 41.2], for which every element is cyclic and which contains a finite number of conjugacy classes.

Note that all these groups are f.g. and simple.

Proposition 4.2. *x If G is a Tarski monster group, then G admits a strongly aperiodic SFT.*

Proof. We take again

$$X = \{x \in \{0,1,2\}^G | \forall g \in G, (gx)_a \neq (gx)_\lambda\}$$

for some $a \neq \lambda$.

The stabilizer of any element of X is not G, thus should be finite. □

In the previous example, the stabilizer of any point is finite, but may be nontrivial. In the two following examples, the stabilizer is trivial.

Proposition 4.3. *If G is a Osin monster group, then G admits a strongly aperiodic SFT s.t the stabilizer of any point is trivial.*

Proof. We take again

$$X = \{x \in \{0,1,2\}^G | \forall g \in G, (gx)_a \neq (gx)_\lambda\}$$

for some $a \neq \lambda$.

Now let $x \in X$. To prove that $Stab(x) = \{\lambda\}$, let $g \in G$, $g \neq \lambda$ s.t. $gx = x$. By definition of the Osin monster group, g is conjugate to a: $hgh^{-1} = a$ for some h, or equivalenty $hg = ah$.

Now $(hx)_a = (hgx)_a = (ahx)_a = (hx)_{a^{-1}a} = (hx)_\lambda$. Thus $(hx)_a = (hx)_\lambda$, contradicting the fact that $x \in X$.

Thus $Stab(x) = \{\lambda\}$. □

Proposition 4.4. *If G is a Ivanov monster group, then G admits a strongly aperiodic SFT s.t the stabilizer of any point is trivial.*

Proof. Let $a_1 \dots a_n$ be distinct representatives of all nontrivial conjugacy classes.

We take

$$X = \{x \in \{0,1,2,\dots 2n\}^G | \forall g \in G \forall i, (gx)_{a_i} \neq (gx)_\lambda\}$$

X is clearly an SFT, and it is clearly nonempty.

Now let $x \in X$. Let $g \in G$, $g \neq \lambda$ s.t. $gx = x$. g is conjugate to a_k for some k. We then repeat the arguments of the previous proof. □

5 On a Construction of Kari

5.1 Definitions

Kari provided a way in [Kar07] to convert a piecewise affine map into a tileset simulating it. We give here the relevant definitions. First, we introduce a formalism for Wang tiles that will be easier to deal with.

Definition 5.1. *Let G be a f.g. group with a set S of generators.*

A set of Wang tiles over G is a tuple $(C, (\phi_h)_{h\in S}, (\psi_h)_{h\in S})$ where, for each h, ϕ_h, ψ_h are maps from C to some finite set.

The subshift generated by C is

$$\begin{aligned} X_C &= \{x \in C^G | \forall g \in G, \forall h \in S, \phi_h(x_g) = \psi(x_{gh^{-1}})\} \\ &= \{x \in C^G | \forall g \in G, \forall h \in S, \phi_h((gx)_e) = \psi((gx)_{h^{-1}})\} \end{aligned}$$

(The last definition proves it is indeed a subshift, and in fact a subshift of finite type). If G has one generator (in particular if $G = \mathbb{Z} = \langle 1 \rangle$), we will write ϕ and ψ instead of ϕ_1 and ψ_1.

Definition 5.2. *Let $cont : \{0,1\}^{\mathbb{Z}} \to [0,1]$ defined by $cont(x) = \limsup_n \frac{\sum_{i\in[-n,n]} x_i}{2n+1}$ and $disc : [0,1] \to \{0,1\}^{\mathbb{Z}}$ defined by $disc(y)_n = \lfloor (n+1)x \rfloor - \lfloor nx \rfloor$.*

Remark that $cont(disc)(y) = y$.

Theorem 3 ([Kar07])**.** *Let a, b be rational numbers and $f(x) = ax + b$.*

Then there exists a set of Wang tiles (C, ϕ, ψ) over $\mathbb{Z}$ and two maps out, in from C to $\{0,1\}$ so that the two following properties hold

- *For any configuration x of X_C, $f(cont(in(x)) = cont(out(x))$*
- *For any $y \in [0,1]$ so that $f(y) \in [0,1]$, there exists a configuration x of C_G so that $in(x) = disc(y)$ and $out(x) = disc(f(y))$*

C is usually seen as a set of Wang tiles over $\mathbb{Z}^2$ rather than $\mathbb{Z}$ but this formalism is better for our purpose.

Two examples are given in Figure 1.

Corollary 5.3. *Let $f_1 \dots f_k$ be a finite family of affine maps with rational coordinates.*

Then there exists a set of Wang tiles (C, ϕ, ψ) over $\mathbb{Z}$ and maps in et $(out_i)_{1\le i\le k}$ from C to $\{0,1\}$ so that the two following properties hold

- *For any configuration x of X_C, $f_i(cont(in(x)) = cont(out_i(x))$*
- *For any $y \in [0,1]$ so that $f_i(y) \in [0,1]$ for all i, there exists a configuration x of C_G so that $in(x) = disc(y)$ and $out_i(x) = disc(f_i(y))$*

Proof. let (C^i, ϕ^i, ψ^i) be the set of Wang tiles over $\mathbb{Z}$ corresponding to f_i, with maps out^i and in^i.

Let $C = \{y \in \prod C^i | \exists x \in \{0,1\}, \forall i, in^i(y^i) = x\}$. Let p^i denote the projection from C to C^i and define

$$\phi = \prod (\phi^i \circ p^i)$$

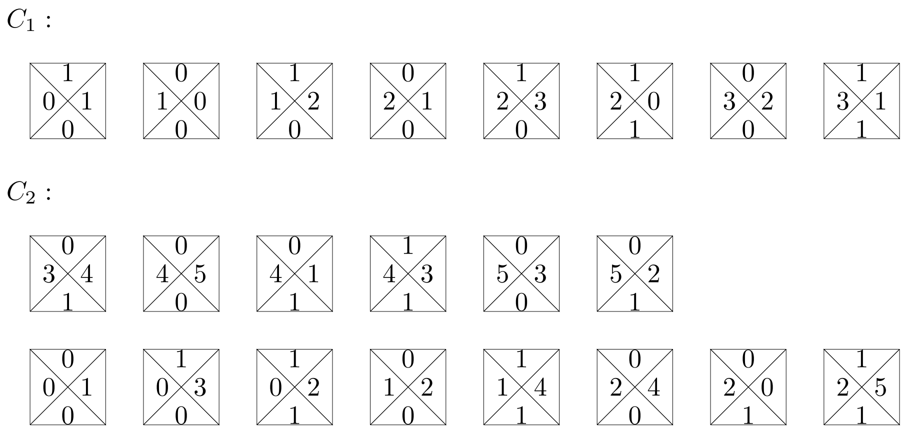

Figure 1: Two set of Wang tiles corresponding respectively to the maps $f(x) = (2x-1)/3$ and $f(x) = (4x+1)/3$. The colors on each tile $c \in C$ on east,west,north,south represent respectively $\phi(c), \psi(c), in(c), out(c)$.

$$\psi = \prod(\psi^i \circ p^i)$$
$$in = in^1 \circ p^1 = in^2 \circ p^2 = \cdots = in^k \circ p^k$$
$$out_i = out^i \circ p^i$$

It is clear that C satisfies the desired properties. ☐

Corollary 5.4. *Theorem 5 still holds when f is a piecewise affine rational homeomorphism from $[0,1]_{/0\sim 1}$. As a consequence, the previous corollary also holds for a finite familiy of piecewise affine rational homeomorphisms.*

Let's define precisely what we mean by a piecewise affine rational homeomorphism from $[0,1]_{/0\sim 1}$ to $[0,1]_{/0\sim 1}$.

We first define a relation $\equiv$: $x \equiv y$ if $x = y$ or $\{x,y\} = \{0,1\}$. Then a piecewise affine homeomorphism from $[0,1]_{/0\sim 1}$ to $[0,1]_{/0\sim 1}$ is given by a finite family $[p_i, p_{i+1}]_{i<n}$ of n intervals with rational coordinates, with $p_0 = 0$ and $p_n = 1$, and a family of affine maps $f_i(x) = a_i x + b$ so that

- f_i and f_{i+1} agrees on their common boundary:
$$\forall i \in \mathbb{Z}/n\mathbb{Z}, f_i(p_{i+1}) \sim f_{(i+1)}(p_{i+1})$$
- $f = \cup_i f_i$ is injective: $f(x) \sim f(y) \implies x \sim y$

These properties imply that f is invertible, and its inverse is still piecewise affine

Proof. We give a proof of this easy result to prepare for another proof later on.

Let $\mathcal{F}$ be the class of relations on $[0,1] \times [0,1]$ for which the theorem is true. $\mathcal{F}$ contains all rational affine maps.

It is clear from the formalism that if f and g are in $\mathcal{F}$, then $f \cup g \in \mathcal{F}$, by taking $(C \cup C', \phi \cup \phi', \psi \cup \psi)$ once the range of ϕ and ϕ' have been made disjoint. In the same way, we may prove $f \circ g \in \mathcal{F}$ and $f; g \in \mathcal{F}$, where $f \circ g(x) = f(g(x))$ and $f; g(x) = g(f(x))$.

Let $i \in \{0 \dots n-1\}$. The function f_i is the composition of 5 functions in $\mathcal{F}$:

- $g_1 = id \cup (x \to x-1) \cup (x \to x+1)$ (g_1 is exactly the relation $\equiv$)
- $g_2 = (x \to p_{i+1} - x) \circ (x \to p_{i+1} - x)$ ($g_2(x) = x$, defined on $[0, p_{i+1}]$)
- $g_3 = (x \to x + p_i) \circ (x \to x - p_i)$ ($g_2(x) = x$, defined on $[p_i, 1]$)
- $g_4 = x \to a_i x + b_i$
- $g_5 = g_1$

Hence $f_i \in \mathcal{F}$, and $f \in \mathcal{F}$.

$\square$

Kari first used this construction [Kar96] to obtain a aperiodic SFT of $\mathbb{Z}^2$: Start with a piecewise rational homeomorphism f with no periodic points (f should be indeed over $[0,1]_{/0\sim 1}$ and not over $[0,1]$ for this to work, as any continuous map from $[0,1]$ to $[0,1]$ has a fixed point by the intermediate value theorem). For example, take

$$f(x) = \begin{cases} (2x-1)/3 & \text{if} \quad 1/2 \leq x \leq 1 \\ (4x+1)/3 & \text{if} \quad 0 \leq x \leq 1/2 \end{cases}$$

Take the set of Wang tiles over $\mathbb{Z}$ given by the theorem, and consider it as a set of Wang tiles over $\mathbb{Z} \times \mathbb{Z}$ by mapping *in* and *out* to $\phi_{(0,1)}$ and $\psi_{(0,1)}$. For the specific function f above, we obtain the set of 22 tiles presented in Fig. 1 (where horizontal colors corresponding to different sets are supposed to be distinct). Then it is easy to see that this indeed gives a SFT over $\mathbb{Z}^2$ with no periodic points. The same construction can be refined [Kar96] to obtain aperiodic tilesets with fewer tiles, but that is not our purpose here.

The important point is that this construction may be easily generalized: There is no need to tile $\mathbb{Z} \times \mathbb{Z}$, we may use the exact same idea to obtain a SFT over $\mathbb{Z} \times G$ for some groups G.

Definition 5.5. *A f.g. group G is PA-recognizable iff there exists a finite set $\mathcal{F}$ of piecewise affine rational homeomorphisms of $[0,1]_{/0\sim 1}$ so that*

- *(A) The group generated by the homeomorphisms is isomorphic to G*
- *(B) For any $t \in [0,1]_{/0\sim 1}$, if $gf(t) = f(t)$ for all f, then $g = e$*

Note that by the property (A) every PA-recognizable group has decidable word problem.

Theorem 4. *If G is PA-recognizable and infinite, there exists a SFT over $\mathbb{Z}\times G$ which is strongly aperiodic.*

Proof. Let S be a set of generators for G. Let $(f_h)_{h\in S}$ be generators for G as a group of piecewise affine maps. And consider the set of Wang tiles (C, ϕ, ψ) and maps in, out_h, corresponding to them by Corollaries 5.3 and 5.4.

Now we look at the set of Wang tiles $(C', (\phi_i), (\psi_i))$ over $\mathbb{Z}\times G$ (with generators 1 and S) defined by $C' = C$ and:

$$\phi_1 = \phi$$
$$\psi_1 = \psi$$
$$\psi_h = in$$
$$\phi_h = out_h$$

We will prove that (a) $X_{C'}$ mimics the behaviour of the piecewise affine maps and (b) gives a strongly aperiodic SFT.

For an element $x \in X_{C'}$, $g \in G$, let $x_g : \mathbb{Z} \to C$ where $x_g(n) = x_{(n,g)}$. Now let $z_g = cont(in(x_g))$. Note that $x_g \in X_C$.

Note that by definition, for any h,

$$\begin{aligned} f_h(z_g) &= f_h(cont(in(x_g))) \\ &= cont(out_h(x_g)) \\ &= cont(\phi_h(x_g)) \\ &= cont(\psi_h(x_{gh^{-1}})) \\ &= cont(in(x_{gh^{-1}})) \\ &= z_{gh^{-1}} \end{aligned}$$

This implies that for any $g = g_1 \dots g_k$, we have $z_{(g_1\dots g_k)^{-1}} = f_{g_1}(f_{g_2}\dots f_{g_k}(z_\lambda)\dots)$. That is, for all $g \in G$, $z_{g^{-1}} = f_g(z_\lambda)$.

Now, by the second part of theorem 5, for any collection $(z_g)_{g\in G}$ that satisfy $z_{g^{-1}} = f_g(z_\lambda)$, there exists a corresponding configuration in $X_{C'}$. This proves that the SFT is nonempty, by starting e.g. with $z_{g^{-1}} = f_g(0)$.

Now let $x \in X_{C'}$ and $(n,h) \in \mathbb{Z}\times G$ be in the stabilizer of x, that is for all $(m,g) \in \mathbb{Z}\times G$, we have $((n,h)x)_{(m,g)} = x_{(m,g)}$. This implies that $z_{gh^{-1}} = z_g$ for all g. Hence, $f_h f_g(z_\lambda) = f_g(z_\lambda)$ for all g. By PA-recognizability, this implies $h = \lambda$.

It remains to prove that $n = 0$. If $n \neq 0$, this means that for each g, the word x_g is periodic of period n. There are finitely many periodic words of length n, which means that z_g will take only finitely many values: $z_g \in Z$ for some finite set Z, which is closed under all maps f_h.

Then each element of G acts as a permutation on Z. Furthermore, by PA-recognizability, any element of G that acts like the identity on Z must be equal to the identity. This implies that any element of G is identified by the permutation of Z it induces, hence that G is finite. ∎

5.2 Applications

Proposition 5.6. *$\mathbb{Z}$ is PA-recognizable. Hence $\mathbb{Z}\times\mathbb{Z}$ admits a strongly aperiodic subshift of finite type*

Proof. The function f seen previously provides a proof.

$$f(x) = \begin{cases} (2x-1)/3 & \text{if} \quad 1/2 \leq x \leq 1 \\ (4x+1)/3 & \text{if} \quad 0 \leq x \leq 1/2 \end{cases}$$

To understand better what f does, we will look at $f' = hfh^{-1}$ where $h(x) = x+1$. Then it is easy to see that

$$f'(x) = \begin{cases} 2x/3 & \text{if} \quad 3/2 \leq x \leq 2 \\ 4x/3 & \text{if} \quad 1 \leq x \leq 3/2 \end{cases}$$

from which it is easy to see that the orbit of f is infinite, (hence the group generated by f is isomorphic to $\mathbb{Z}$), and that if $f^n(t) = t$ for some n and some t, then $n = 0$ (hence property (B)). Therefore G is PA-recognizable. ☐

Proposition 5.7. *Thompson group T is PA-recognizable. Hence $\mathbb{Z}\times T$ admits a strongly aperiodic subshift of finite type*

Proof. T is the quintessential PA-recognizable group: It is formally the subgroup of all piecewise affine maps of $[0,1]_{/0\sim 1}$ where each affine map has dyadic coordinates and positive slope.

T is indeed finitely generated, more precisely it is generated by the three following functions [CFP96]:

$$a(x) = \begin{cases} x/2 & 0 \leq x \leq 1/2 \\ x-1/4 & 1/2 \leq x \leq 3/4 \\ 2x-1 & 3/4 \leq x \leq 1 \end{cases} \quad b(x) = \begin{cases} x & 0 \leq x \leq 1/2 \\ x/2+1/4 & 1/2 \leq x \leq 3/4 \\ x-1/8 & 3/4 \leq x \leq 7/8 \\ 2x-1 & 7/8 \leq x \leq 1 \end{cases}$$

$$c(x) = \begin{cases} x/2+3/4 & 0 \leq x \leq 1/2 \\ 2x-1 & 1/2 \leq x \leq 3/4 \\ x-1/4 & 3/4 \leq x \leq 1 \end{cases}$$

From the definition of T, it is easy to see that the orbit of any $z \in [0,1]$ is dense, hence property (B) is true, and T is PA-recognizable.

☐

Note that it is not clear if Thompson group F is PA-recognizable: F is the subgroup of T generated by a and b, and fixes 0: As a consequence, this particular representation does not satisfy property (B). Whether another representation of Thompson group F exists with this property is open.

Proposition 5.8. *$PSL_2(\mathbb{Z})$ is PA-recognizable. Hence $\mathbb{Z}\times PSL_2(\mathbb{Z})$ admits a strongly aperiodic subshift of finite type.*

Proof. $PSL_2(\mathbb{Z})$ is the subgroup of T generated by a and c, see for example [Fos11]. To see that this representation satisfies property (B), remark that this action is conjugated by the Minkowski question mark symbol (ibid.) to the

action of $PSL_2(\mathbb{Z})$ over the projective line $\mathbb{R} \cup \{\infty\}$. From this point of view, it is clear that any element of $PSL_2(\mathbb{Z})$ different from the identity fixes at most two points. Hence if $gf(t) = f(t)$ for all t, then $g = \lambda$.

For this particular group, it is actually easy to work out all details and produce a concrete aperiodic set of Wang tiles, represented in Fig 2. It is obtained by taking $d = a$ and $e = ac$ as generators (rather than a and c) and looking at them as acting on $[0,2]_{0\sim 2}$ (rather than $[0,1]_{0\sim 1}$) by the formulas:

$$d(x) = \begin{cases} x/2 & 0 \le x \le 1 \\ x - 1/2 & 1 \le x \le 3/2 \\ 2x - 2 & 3/2 \le x \le 2 \end{cases} \quad e(x) = \begin{cases} x+1 & 0 \le x \le 1 \\ x-1 & 1 \le x \le 2 \end{cases}$$

(Of course, such details may also be provided for Thompson group T. However the presence of the generator b produced an set of tiles too large to be depicted here.) $\square$

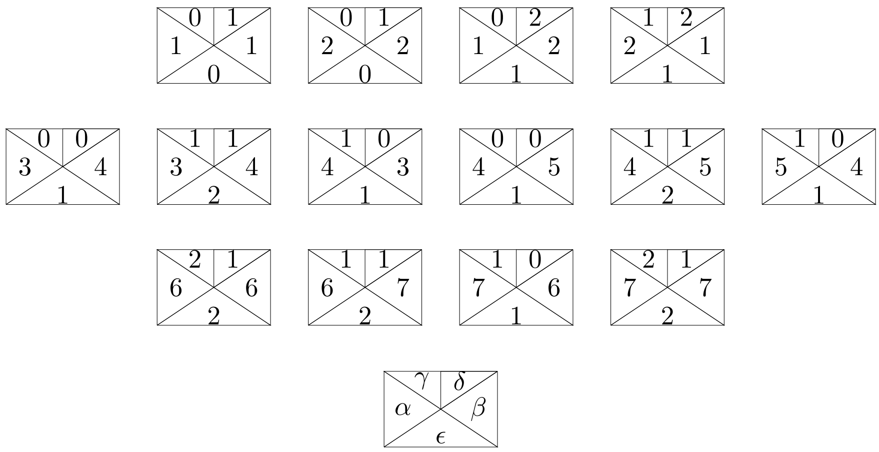

Figure 2: A strongly aperiodic set of 14 Wang tiles over $\mathbb{Z} \times PSL_2(\mathbb{Z})$, where $PSL_2(\mathbb{Z})$ is generated by $d = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The rules are as follows: Let x be the tile in position (n, g). Then the tile y in position $(n+1, g)$ must satisfy $y_\alpha = x_\beta$, the tile y in position (n, gd) must satisfy $y_\epsilon = x_\gamma$, the tile y in position (n, ge) must satisfy $y_\epsilon = x_\delta$.

5.3 Generalizations

The construction of Kari works for more than piecewise affine homeomorphisms of $[0,1]$. It works for any partial piecewise affine map from $[0,1]^d$ to its image.

Theorem 5 ([Kar07]). *Let $A \in M_{m\times n}(\mathbb{Q})$ be a (possibly non square) matrix with rational coefficients, $b \in \mathbb{Q}^m$ a rational vector and $f(x) = Ax + b$*

Then there exists a set of Wang tiles (C, ϕ, ψ) over $\mathbb{Z}$ (generated by 1) and two maps out, in from C to $\{0,1\}^m$ and $\{0,1\}^n$ so that the two following properties hold

- *For any configuration x of X_C, $f(cont_n(in(x)) = cont_m(out(x))$*
- *For any $y \in [0,1]^n$ so that $f(y) \in [0,1]^m$, there exists a configuration x of C_G so that $in(x) = disc_n(y)$ and $out(x) = disc_m(f(y))$*

where $disc_i$ and $cont_i$ are the natural i-dimensional analogues of disc and cont

Now we will be able to prove a theorem similar to the previous one for a larger class of maps (hence a larger class of groups). There are three directions in which we can go:

- Go to higher dimensions
- Look at piecewise affine maps defined on compact subsets of $\mathbb{R}^d$ different from $[0,1]^d$.
- Consider other identifications than $0 \sim 1$

In the following we will not use the full possible generalisation, and will not identify any points in our sets. This will be sufficient for the applications and already relatively painful to define. However, this means that the next definition will not encompass PA-recognizable groups.

Definition 5.9. *Let $\mathcal{F} = \{f_i : B_i \mapsto B'_i, i = 1 \dots k\}$ be a finite set of piecewise affine rational homeomorphisms, where each B_i and B'_i is a finite union of bounded rational polytopes of $\mathbb{R}^n$.*

Let $S_\mathcal{F}$ be the closure of the set f_i and f_i^{-1} under composition. Each element of $S_\mathcal{F}$ is a piecewise affine homeomorphism, whose domain is the union of finitely many bounded rational polytopes, and may possibly be empty.

Let $T_\mathcal{F}$ be the common domain of all functions in $S_\mathcal{F}$.

Then the group *$G_\mathcal{F}$ generated by $\mathcal{F}$ is the group $\{f_{|T_\mathcal{F}}, f \in S_\mathcal{F}\}$.*

Definition 5.10. *A f.g. group G is PA'-recognizable iff there exists a finite set $\mathcal{F}$ of piecewise affine rational homeomorphisms so that*

- *(A) G is isomorphic to $G_\mathcal{F}$.*
- *(B) For any $t \in T_\mathcal{F}$, if $gf(t) = f(t)$ for all f, then $g = \lambda$*

Note that $T_\mathcal{F}$ might not be computable in general. In particular, it is not clear that any PA'-recognizable has decidable word problem.

Theorem 6. *If G is PA'-recognizable, then the complement of the word problem on G is recognizable. In particular, if G is recursively presented, the word problem on G is decidable*

Proof. We assume that $G \neq \{\lambda\}$, hence $T_{\mathcal{F}} \neq \emptyset$.

Let g be an element of G, given by composition of some piecewise affine maps. Let

$$D = \{t | \forall f \in S_{\mathcal{F}}, f(t) \text{ is defined and } g(f(t)) = f(t)\}$$

Note that $D \subseteq T_{\mathcal{F}}$. Furthermore, $g \neq \lambda$ iff $D = \emptyset$ by property (B).

This gives a semi algorithm to decide if $g \neq \lambda$. □

Theorem 7. *If G is PA'-recognizable, $\mathbb{Z}\times G$ admits a strongly aperiodic subshift of finite type.*

Proof. Same proof as before. □

Here a few applications:

Proposition 5.11. *$\mathbb{Z}$ is PA'-recognizable. Hence $\mathbb{Z}\times\mathbb{Z}$ admits a strongly aperiodic subshift of finite type.*

Proof. Let $A = \{(x,y) \in [-1,1]^2, |x| + |y| \geq 1\}$. A is the union of four bounded polytopes.

Let

$$\begin{array}{cccc} f: & A & \to & f(A) \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

And let $\mathcal{F} = \{f\}$. f is clearly an homeomorphism. Note that f is a rotation of angle $\arccos 3/5$.

Now it is easy to see that $T_{\{f\}} = S_1 = \{(x,y) | x^2 + y^2 = 1\}$, and that $G_{\{f\}}$ is isomorphic to $\mathbb{Z}$. Furthermore, it is also clear that the orbit of any point of $T_{\{f\}}$ is dense in $T_{\{f\}}$, which implies property (B). Hence $\mathbb{Z}$ is PA'-recognizable. □

Proposition 5.12. *Any finitely generated subgroup G of rational matrices of a compact matrix group is PA'-recognizable. Hence $\mathbb{Z}\times G$ admits a strongly aperiodic subshift of finite type.*

Proof. We assume familiary with representation theory of linear compact groups, see e.g. [OV90, Chap 3.4]. Let G be such a group, and let $M_1 \dots M_n$ be the matrices of size $k \times k$ that generate G. Using elementary linear algebra we may suppose there exists a rational vector $y \in \mathbb{R}^k$ so that $gy = y \to g = \lambda$, and Gy spans $\mathbb{R}^k$.

Now, as G is a subgroup of a compact group, we can define a scalar product so that all matrices of G are unitary. Let $\mathbb{R}^k = V^1 \oplus V^2 \cdots \oplus V^p$ be a decomposition of $\mathbb{R}^k$ into orthogonal (for this scalar product) irreducible G-invariant vector spaces, that is $GV^i = V^i$ and no proper nonzero subspace of V^i is G-invariant. This is possible as G is a subgroup of a compact group hence completely reducible. Note that the vector spaces V^i might not have rational bases.

Let P^i be the orthogonal projection onto V^i. For a vector x, let $x^i = P^i x$, so that $x = \sum_i x^i$. For a matrix $g \in G$, let $g^i : V^i \to V^i$ be the restriction of g to V^i, so that $gx = \sum_i g^i x^i$.

Recall there is y so that $gy = y$ for $g \in G$ iff $g = \lambda$. As Gy spans $\mathbb{R}^k$, $y^i \neq 0$ for all i.

Let $i \in \{1 \dots p\}$. Let W^i be the topological closure of Gy^i. As y^i is nonzero and V^i is G-invariant, $W^i \subseteq V^i$ and is faraway from zero. That is, there exists constants $r^i, R^i > 0$ so that for all $y \in W^i$, $|y|_1 > r^i$ and $|y|_1 < R^i$.

Now let

$$T = \{y | \forall i, |P^i y|_1 > r^i \text{ and } |P^i y|_1 < R^i\}$$

and

$$T_0 = \{y | \forall i, |P^i y|_1 > r^i/2 \text{ and } |P^i y|_1 < 2R^i\}$$

Note that T is a polytope with real coordinates. Let T' be an approximation of T as a polytope with rational coordinates, so that $T \subseteq T' \subseteq T_0$.

Now define the maps f_i as restrictions of M_i from T' to $M_i T'$. Let $\mathcal{F}$ be the corresponding set of maps.

We cannot describe $T_{\mathcal{F}}$ exactly, but it is clear that it contains y, as T contains the G-orbit of y. As a consequence, $G_{\mathcal{F}}$ is isomorphic to G.

Now we prove property (B). Start from $t \in T_{\mathcal{F}}$ and $g \in G_{\mathcal{F}}$ so that $gf(t) = f(t)$ for all f.

Let $i \in \{1, \dots, p\}$ and let $t^i = P^i t$ so that $t = \sum_i t^i$. As $t \in T_{\mathcal{F}} \subseteq T' \subset T_0$, we have $t^i \neq 0$. As a consequence, the orbit of Gt^i on W_i spans a nonzero G-invariant subspace of V_i, which is V_i by irreducibility. Now, as $gf(t) = f(t)$ for all f, we conclude that g is the identity on the orbit of Gt^i, hence g is the identity on V^i. As this is true for all i, g is the identity matrix. □

Corollary 5.13. *The free group $\mathbb{F}_2$ is PA'-recognizable. Every finite group is PA'-recognizable.*

Proposition 5.14. *Thompson's group V is PA'-recognizable.*

Proof. V is usually given [CFP96] as the generalization of T to discontinuous maps. However, our maps in the definition need to be continuous, so we will see V as acting on the "middle thirds" Cantor set (As a side note, V is therefore isomorphic to the group of all revertible generalized one-sided shifts [Moo91]).

Let

$$C_3 = \left\{ \sum_{i \geq 1} \frac{\alpha_i}{3^i}, \alpha \in \{0,2\}^{\mathbb{N}^+} \right\}$$

Let a, b, c, π_0 defined on C_3 by:

$$a(x) = \begin{cases} x/3 & 0 \leq x \leq 1/3 \\ x - 4/9 & 2/3 \leq x \leq 7/9 \\ 3x - 2 & 8/9 \leq x \leq 1 \end{cases} \quad b(x) = \begin{cases} x & 0 \leq x \leq 1/3 \\ x/3 + 4/9 & 2/3 \leq x \leq 7/9 \\ x - 4/27 & 8/9 \leq x \leq 25/27 \\ 3x - 2 & 26/27 \leq x \leq 1 \end{cases}$$

$$c(x) = \begin{cases} x/3 + 8/9 & 0 \leq x \leq 1/3 \\ 3x - 2 & 2/3 \leq x \leq 7/9 \\ x - 2/9 & 8/9 \leq x \leq 1 \end{cases} \quad \pi_0(x) = \begin{cases} x/3 + 2/3 & 0 \leq x \leq 1/3 \\ 3x - 2 & 2/3 \leq x \leq 7/9 \\ x & 8/9 \leq x \leq 1 \end{cases}$$

Now our definition does not permit to define a, b, c, π_0 on C_3, as the domain and range of each map should be a finite union of intervals with rational coordinates. So we will define them by the above formulas, but for $x \in [0,1]$ rather than $x \in C_3$. Note that they are already homeomorphisms onto their image.

Let $\mathcal{F} = \{a, b, c, \pi_0\}$. We claim that $T_{\mathcal{F}} = C_3$, which will prove that $G_{\mathcal{F}}$ is indeed isomorphic to V. As before, any orbit is dense, from which property (B) ensues and V will be PA'-recognizable.

It remains to prove that $T_{\mathcal{F}} = C_3$. Note that clearly $C_3 \subseteq T_{\mathcal{F}}$.

First note that

- $Dom(a) = [0, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$
- $Range(a) = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 1]$

Which implies that $T_{\mathcal{F}} \subseteq [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

Now let $x \in T_{\mathcal{F}}$.

- If $0 \leq x \leq 1/9$, then $3x \in T_{\mathcal{F}}$ (apply a^{-1})
- if $2/9 \leq x \leq 1/3$, then $3x \in T_{\mathcal{F}}$ (apply a^{-1}, then c^{-1} then a)
- if $2/3 \leq x \leq 7/9$, then $3x - 2 \in T_{\mathcal{F}}$ (apply c)
- If $8/9 \leq x \leq 1$, then $3x - 2 \in T_{\mathcal{F}}$ (apply a)

This proves inductively that $x \in C_3$. $\square$

Open Problems

This is only one way of generalizing Kari's construction. There are many other ways to generalize it, one of which providing a (weakly) aperiodic SFT on the Baumslag Solitar group, see [AK13].

Here is an interesting open question: The construction uses representations of reals as words in $\{0, 1\}^{\mathbb{Z}}$, can we use a representation in $\{0, 1\}^H$, for some other group H ? This would possibly allow to prove that $H \times G$ has a strongly aperiodic SFT for G PA-recognizable.

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